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Amalgams of p-Groups

GRAHAM HIGMAN

Mathematical Institute, 10 Parks Road, Oxford, England

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I. INTRODUCTION

Suppose that $A \cup B$ is an amalgam, and that A, B are finite p-groups. If $A \cup B$ can be embedded in a finite p-group C there is a homomorphism of the free product F of A and B, with the amalgamation $U = A \cap B$, into C. The kernel of this homomorphism is a normal subgroup of F which intersects A and B trivially, and is therefore a free group.[1] It follows that F is residually a finite p-group, and hence is residually nilpotent. Conversely, since F is generated by a finite number of elements of p-power order, if it is residually nilpotent it is residually a finite p-group, and hence the amalgam $A \cup B$ is embeddable in a finite p-group. The object of this note is to give a simple internal condition on the amalgam for these equivalent conditions to hold.

If H is a subgroup of the finite p-group G, and $(G_i) = (G_0, G_1, \cdots, G_n)$ is a chief series of G, the distinct terms of the series $(H \cap G_0, H \cap G_1, \cdots, H \cap G_n)$ form a chief series of H, which we write $H \cap (G_i)$. Moreover if K is again a subgroup of H,

$$K \cap (G_i) = K \cap (H \cap (G_i)).$$

In particular, if the amalgam $A \cup B$ is embeddable in the finite p-group C, then, choosing any chief series (C_i) of C, we can construct chief series

$$(A_i) = A \cap (C_i) \qquad \text{and} \qquad (B_i) = B \cap (C_i)$$

of A and B such that, if $U = A \cap B$, $U \cap (A_i) = U \cap (B_i)$. Our main object is to show that this necessary condition for embeddability is also sufficient. That is, we shall prove the theorem following, where the equivalence of conditions (i) to (iii) and the fact that these imply (iv) are trivial; the business part is that conversely (iv) implies (iii).

[1] This follows from the theorem of H. Neumann [2], but can also, probably more easily, be proved directly.

THEOREM. *Let $A \cup B$ be an amalgam of finite p-groups A, B with $A \cap B = U$. Let F be the free product of A and B with the amalgamation U. Then the following four statements are equivalent.*

(i) *F is residually nilpotent.*
(ii) *F is residually a finite p-group.*
(iii) *$A \cup B$ is embeddable in a finite p-group.*
(iv) *There exist chief series (A_i) of A and (B_i) of B such that*

$$U \cap (A_i) = U \cap (B_i).$$

COROLLARY 1. *If U is cyclic statements (i) to (iii) hold.*
For in this case U has only one chief series, so (iv) must be true. This answers a question put to me in a letter by G. Baumslag.
Next, if U is a subgroup of a group G, we denote by $\mathrm{Aut}_G(U)$ the image of the normalizer of U in G in its natural homomorphism into the automorphism group of U.

COROLLARY 2. *If U is normal both in A and in B, then (i) to (iii) hold if and only if* $\mathrm{Aut}_A(U)$ *and* $\mathrm{Aut}_B(U)$ *generate a p-group.*
If $A \cup B$ is embeddable in the p-group C, then $\mathrm{Aut}_A(U)$ and $\mathrm{Aut}_B(U)$ are contained in the p-group $\mathrm{Aut}_C(U)$. Conversely, if $\mathrm{Aut}_A(U)$ and $\mathrm{Aut}_B(U)$ are both contained in the p-group H, we form the subgroup HU of the holomorph of U, and lay a chief series of it through U. The portion of this series below U is a chief series of U consisting of subgroups which are normal alike in A and in B, and therefore extensible equally to a chief series of A and a chief series of B. Thus condition (iv) of the theorem holds, and the corollary is proved.

II. EXTENSIONS AND WREATH PRODUCTS

If $A \cup B$ is an amalgam, with $A \cap B = U$, a *subamalgam* of $A \cup B$ is an amalgam $X \cup Y$ with $X \leqslant A$, $Y \leqslant B$ and $X \cap U = Y \cap U$. The subamalgam is *normal* if X is normal in A and Y in B. There then exists a *factor amalgam* $(A/X) \cup (B/Y)$, where UX/X and UY/Y are identified via their natural isomorphisms with $U/U \cap X = U/U \cap Y$. Now if X is a normal subgroup of A, and $A/X \simeq H$, A can be embedded in the wreath product $X \wr H$ [*1*]. It might therefore be supposed that if the normal subamalgam $X \cup Y$ can be embedded in the group T and the factor amalgam $(A/X) \cup (B/Y)$ in H, then $A \cup B$ can be embedded in $T \wr H$. It is easy to give examples to show that in general this is false. However, the proof of the theorem can conveniently be based on a lemma (Lemma 2 below) which says that it is true if in addition we have $U \cap X = U \cap Y$ central both in A and in B.

We first define wreath products and construct the embeddings in the most convenient way. If X, H are groups the set X^H of maps of H into X is a group under coordinate-wise multiplication (i.e., defining fg by $fg(h) = f(h)\,g(h)$) and admits H as a group of automorphisms, by $f^{h_1}(h) = f(h_1 h)$. The wreath product $X \wr H$ is then the subgroup HX^H of the holomorph of X^H, so that its elements are pairs (h, f), h in H, f in X^H with the multiplication rule $(h_1, f_1)(h_2, f_2) = (h_1 h_2, f_1^{h_2} f_2)$.

If θ is a homomorphism of a group A into a group H with kernel X a *countermap* to θ is a map θ^* (not necessarily a homomorphism) of H into A such that $\theta^*\theta$ is an endomorphism of H quâ left $A\theta$-space. That is, such that, for a in A and h in H,

$$(a\theta \cdot h)\,\theta^*\theta = a\theta \cdot h\theta^*\theta. \tag{1}$$

Countermaps always exist, for if the images $h\theta^*$ are chosen arbitrarily for h in a set of representatives of the cosets $A\theta h$, (1) merely prescribes the cosets of X from which the remaining $h\theta^*$ are to be chosen. For fixed a in A an element f_a of X^H is defined by

$$f_a(h) = [(a\theta \cdot h)\,\theta^*]^{-1}\, a \cdot h\theta^*,$$

since by (1) the right hand side is in the kernel of θ. It is easily verified that the map $a \to (a\theta, f_a)$ embeds A in $X \wr H$. We describe this as a *standard embedding* of A in $X \wr H$.

If now U is a subgroup of A, the restriction of θ to U has kernel $U \cap X$, and so gives rise to standard embeddings of U in $(U \cap X) \wr H$. But there is a natural embedding of $(U \cap X) \wr H$ in $X \wr H$, so we may think of them as embedding U in $X \wr H$.

LEMMA 1. *Let U be a subgroup of A and θ a homomorphism of A into H. Given any standard embedding $u \to (u\theta, f_u)$ of U in $X \wr H$ we can find a standard embedding $a \to (a\theta, g_a)$ of A in $X \wr H$ such that for u in U, h in H, $f_u(h)$ and $g_u(h)$ are conjugate in A.*

Let the standard embedding of U be made with the countermap φ^* to the restriction of θ to U; and let θ^* be a countermap to θ. Define the map ω of H to A by

$$h\theta^* = h\varphi^* \cdot h\omega.$$

For u in U and h in H we have

$$(u\theta \cdot h)\,\varphi^*\theta = u\theta \cdot h\varphi^*\theta$$

and

$$(u\theta \cdot h)\,\theta^*\theta = u\theta \cdot h\theta^*\theta,$$

whence

$$(u\theta \cdot h)\,\omega\theta = h\omega\theta.$$

That is, $\omega\theta$ is constant on the cosets $U\theta h$. Now multiplying θ^* by any function from H to X yields another countermap θ_1^*. Clearly we can choose the function from H to X so that, if $h\theta_1^* = h\varphi^* h\omega_1$, not only $\omega_1\theta$ but even ω_1 is constant on the cosets $U\theta h$. If $a \to (a\theta, g_a)$ is the standard embedding of A in $X \wr H$ formed with the countermap θ_1^* we have, for u in U, h in H

$$\begin{aligned} g_u(h) &= [(u\theta \cdot h)\,\theta_1^*]^{-1}\, u \cdot h\theta_1^* \\ &= [(u\theta \cdot h)\,\omega_1]^{-1}\,[(u\theta \cdot h)\,\varphi^*]^{-1}\, u \cdot h\varphi^* \cdot h\omega_1 \\ &= (h\omega_1)^{-1} f_u(h)\, h\omega_1 \end{aligned}$$

as required.

COROLLARY. *If $U \cap X$ is contained in the center of A any a standard embedding of U in $X \wr H$ can be extended to an embedding of A in $X \wr H$.*

For $f_u(h) \in U \cap X$, so that then $g_u = f_u$.

LEMMA 2. *Let $X \cup Y$ be a normal subamalgam of the amalgam $A \cup B$, where $A \cap B = U$, and suppose that $Z = U \cap X = U \cap Y$ is central both in A and in B. If $X \cup Y$ can be embedded in T and $(A/X) \cup (B/Y)$ in H, $A \cup B$ can be embedded in $T \wr H$.*

The embedding of $(A/X) \cup (B/Y)$ in H gives homomorphisms $\theta : A \to H$ and $\varphi : B \to H$ which agree on U. By the corollary to Lemma 1, a standard embedding of U in $Z \wr H$ can be extended to embeddings of A in $X \wr H$ and of B in $Y \wr H$, and hence of both in $T \wr H$, as required.

III. PROOF OF THEOREM, AND REMARKS

We now prove that (iv) of the theorem implies (iii), using induction on the product of the orders of A and B, since if these are both 1 the theorem is trivial.

Then A, B have designated chief series, since we are assuming (iv), and we take A_1, B_1 to be the groups of order p in these series. If A_1 is not contained in U, we put $A_1 = X$, and $1 = Y$, and if B_1 is not contained in U, we put $1 = X$, $B_1 = Y$. (If neither A_1 nor B_1 is contained in U we can put alternatively $A_1 = X$, $B_1 = Y$.) The hypotheses of Lemma 2 hold since $Z = 1$; and we can take T to be the direct product $X \times Y$. Moreover, the amalgam $(A/X) \cup (B/Y)$ satisfies (iv) taking as designated chief series the series

(A_i/X) and (B_i/Y) where (A_i), (B_i) are the designated series in A, B. Thus by the induction hypothesis we can take H to be a p-group. Then $T \propto H$ is a p-group, and we are home. If, finally, both A_1 and B_1 are contained in U then $A_1 = B_1$, since

$$U \cap (A_i) = U \cap (B_i),$$

and we take $X = Y = A_1 = B_1$. Again the conditions of Lemma 2 hold, and the inductive hypothesis enables us to take H to be a p-group. As T we can, of course, take X. Thus the theorem is completely proved.

The argument, as may be verified easily, in fact proves slightly more. Namely, if the amalgam $A \cup B$ satisfies (iv) it can be embedded in a finite p-group C with a chief series (C_i) such that

$$(A_i) = A \cap (C_i) \qquad \text{and} \qquad (B_i) = B \cap (C_i).$$

It is useful to notice this when considering the possibility of extending the theorem to amalgams of more than two groups. Two cases arise.

First we can consider an amalgam $A_1 \cup A_2 \cup \cdots \cup A_k$ such that Schreier's theorem guarantees that its free product exists. That is, the amalgam is the union $B \cup C$ of subamalgams

$$B = A_1 \cup A_2 \cup \cdots \cup A_s, \qquad C = A_{s+1} \cup \cdots \cup A_k$$

of the same kind, with $B \cap C = A_s \cap A_{s+1}$. In this case it is easy to see, by induction on k, and using the above remark, that a condition analogous to (iv) ensures the embeddability of the amalgam in a finite p-group.

In general however, as may be seen by construction of examples, the analogue of (iv) does not ensure embeddability in a group at all; and it seems highly unlikely that it could ensure that an amalgam embeddable, as it were by chance, in some group or other is necessarily embeddable in a finite p-group.

REFERENCES

1. KALOUJNINE, L. AND KRASNER, M. Produit complet des groupes de permutation et le problème d'extension des groupes III. *Acta Sci. Math. Szeged* **14** (1951), 69-82.
2. NEUMANN, H. Generalized free products with amalgamated subgroups II. *Am. J. Math.* **31** (1949), 491-540.

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